Noninvertible Selection Rule for Solitons

Yuya Tanizaki (Yukawa Institute, Kyoto U.)

Based on arXiv:2210.13780, 2307.00939
with Shi Chen (陈实) at U. Minnesota

If several solitons have different dimensions,

their selection rule becomes non-invertible symmetry.

Outline

1. Introduction
2. 4d $\mathbb{CP}^1$ σ-model & Hopfion
3. Mathematical structure
4. Summary

[Related works : P.S. Hsin 2212.08608,
S.D. Pace 2308.05703, Pace, Zhu, Beaudry, Wen 2310.08554]

When (usual) symmetry is spontaneously broken

$$G \xrightarrow{SSB} H,$$

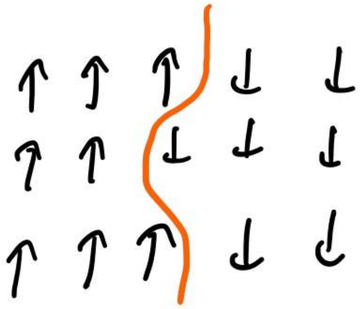
the low-energy effective theory is given by a non-linear σ-model:

$$\sigma : \text{Spacetime} \longrightarrow G/H$$

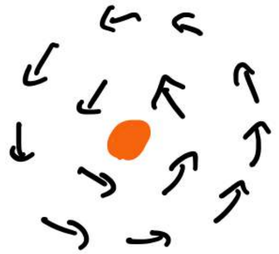
$$Z = \int \mathcal{D}\sigma \exp\left(-\frac{1}{g^2}\int |d\sigma|^2 + (\cdots)\right).$$

Small fluctuations of σ : Nambu-Goldstone modes

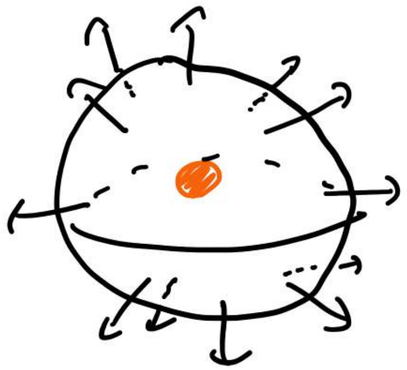
Large fluctuation : Topological Solitons



domain wall



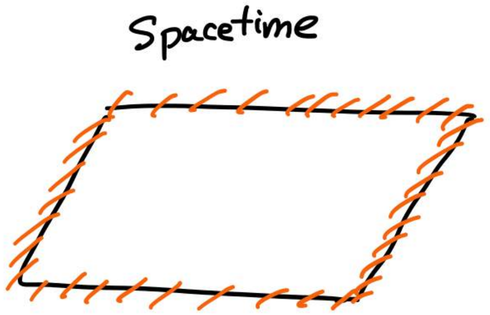
vortex



hedgehog

Topological Stability and Homotopies

(Mermin '79 Rev. Mod. Phys.)



$\xrightarrow{\sigma}$ $\mathcal{M} (= G/H)$.

To find finite action/energy (density) $\int |d\sigma|^2 < \infty$, its convenient to identify ∞'s of $\mathbb{R}^n$:

$$\mathbb{R}^n \cup \{\infty\} \simeq S^n \xrightarrow{\sigma} \mathcal{M}.$$

$\Rightarrow$ Topoligcal solitons are classified by homotopies of the target space

$$\pi_n(\mathcal{M}).$$

Recall that

Conservation Law $\Longleftrightarrow$ Symmetry,

we should be able to understand this topological conservation law as the symmetry of the σ-model.

Conventional Wisdom: Solitonic Sym. $\cong$ Hom $(\pi_n(\mathcal{M}), U(1))$.

↳ Is this always true?

Why did we ask this question?

Some mysterious behavior of $SO(N_c)$ QCD

$$\mathcal{L} = \frac{1}{g^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^{N_f} \bar{\psi}_i \bar{\sigma}^\mu (\partial_\mu + i a_\mu)\psi_i .$$

Flavor symmetry : $SU(N_f) \xrightarrow{\text{SSB}} SO(N_f)$

$\Rightarrow \frac{SU(N_f)}{SO(N_f)}$ σ - model : effective theory of "pions".

Baryons $\varepsilon^{i_1 \cdots i_{N_c}} \psi_{i_1 j} \cdots \psi_{i_{N_c} j}$ = Skyrmions at $N_c = \infty$

For $SO(N_c)$ QCD, baryons can annihilate in pairs

	$N_f = 2$	$N_f = 3$	$N_f \geq 4$
π_2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_3	$\mathbb{Z}$	$\mathbb{Z}_4$	$\mathbb{Z}_2$

Skyrmions do not annihilate in pairs: They have richer spectrum. [Witten '83, Imoto, Sakai, Sugimoto '09]

→ Let us revisit the conservation law of solitons.

$(3+1)$d $\mathbb{C}P^1$ σ-model

Assume some $(3+1)$d quantum systems have SSB

$$SU(2) \xrightarrow{\text{SSB}} U(1) \Rightarrow \mathbb{C}P^1 \simeq \frac{SU(2)}{U(1)} \ \sigma\text{-model}.$$

Lagrangian:

$$\begin{cases} \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : \mathbb{C}^2\text{-valued scalar field with } |\vec{z}|^2 = 1. \\ a = a_\mu dx^\mu : \text{(auxiliary) } U(1) \text{ gauge field} \end{cases}$$

$$\mathcal{L} = \frac{1}{g^2} |(\partial_\mu + i a_\mu)\vec{z}|^2.$$

This $U(1)$ gauge field a is auxiliary because its EoM can be solved as

$$a = i \vec{z}^\dagger \cdot d\vec{z}$$

Homotopy of $\mathbb{C}P^1$ $(\simeq S^2)$:

$$\pi_1(\mathbb{C}P^1) \simeq 0 \ , \ \underline{\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}} \ , \ \underline{\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}}$$

"magnetic skyrmion"
"monopole"

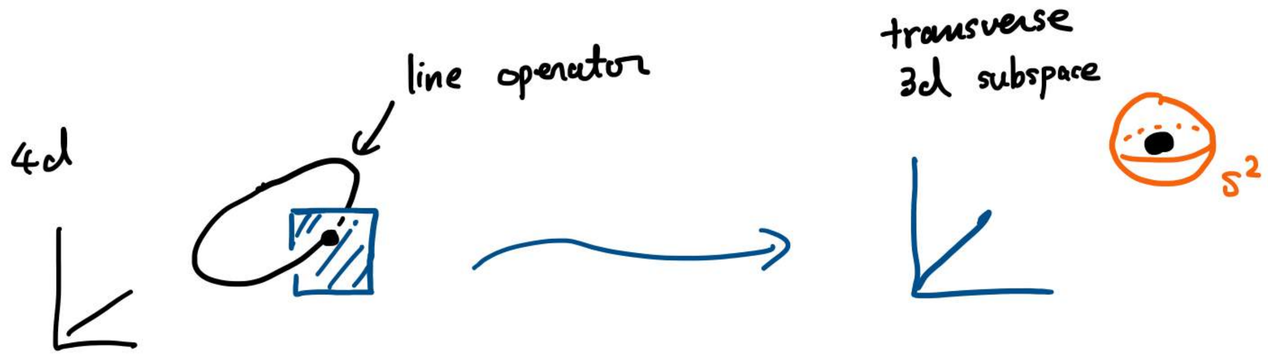
"Hopfion"

Vortex Soliton $\pi_2(\mathbb{CP}^1)$

$\pi_2(\mathbb{CP}^1) \simeq \mathbb{Z}$ has the Noether current

$$j = \frac{1}{2\pi} da,$$

and it gives U(1) 1-form symmetry.



$$\underbrace{\int_{S^2} \frac{da}{2\pi} \in \mathbb{Z}}_{\text{Dirac quantization}} \simeq \pi_2(\mathbb{CP}^1)$$

We denote this line operator as $V_n(L)$ with $n = \int_{S^2} \frac{da}{2\pi}$.

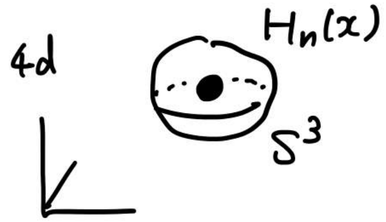
(We'll see that there is a finer classification for the vortex operators $V_n(L)$)

Hopfion $\pi_3(\mathbb{C}P^1)$

$\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ follows from the Hopf fibration $S^1 \to S^3 \to S^2$, and the corresponding solitons are known as "Hopfion" (or "Hopf soliton").

$\pi_3(\mathbb{C}P^1)$ is measured by the Hopf number [Wilczek, Zee '83]

$$\frac{1}{4\pi}\int_{S^3} a\,da \quad \in \pi\,\mathbb{Z}$$



(* For general $U(1)$ gauge fields, the Chern-Simons form $\int \frac{1}{4\pi} a\,da$ can take arbitrary numbers.
Here, since a is an "auxiliary" field ($a = i\vec{z}^{\dagger} d\vec{z}$),
$d\left(\frac{1}{4\pi} a\,da\right) = \frac{1}{4\pi}(da)^2 = 0 \Rightarrow \int \frac{1}{4\pi} a\,da$ becomes quantized.)

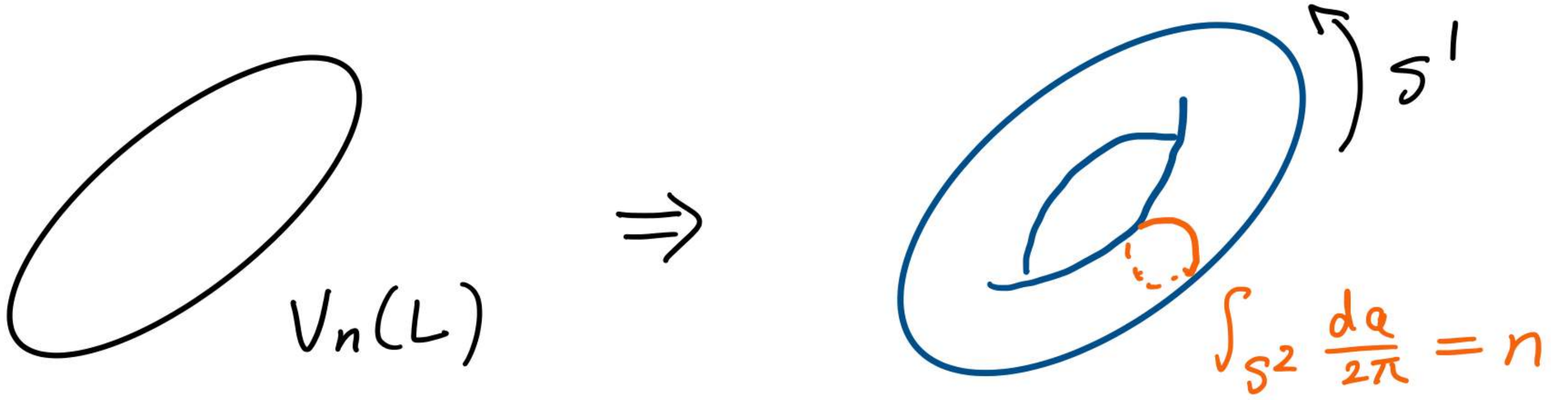
Unlike the case of $\pi_2(\mathbb{C}P^1)$, however, the integrand

$$j_{\text{Hopf}} := \frac{1}{4\pi^2}\, a\,da$$

is not gauge invariant.

$U(1)$ without Noether current ?

More on Vortex Operators $V_{n,k}(L)$



Let's classify the $\mathbb{C}P^1$ configurations

$$S^2 \times S^1 \xrightarrow{\sigma} \mathbb{C}P^1$$

up to homotopy for a given monopole charge $\int_{S^2} \frac{da}{2\pi} = n$.

In particular, what is the possible value of

$$\int_{S^2 \times S^1} \frac{a\,da}{4\pi^2} \quad (=: k) \quad ? \quad \Rightarrow V_{n,k}(L).$$

$\mathbb{Z}_{2n}$ classification : $V_{n,k+2n}(L) \cong V_{n,k}(L)$

Let's try to evaluate the Hopfion number on $S^2 \times S^1$:

$$k = \int_{S^2 \times S^1} \frac{a da}{4\pi^2}$$

Under the large $U(1)$ gauge transformation $a \longmapsto a + \epsilon^{(1)}$ along S^1,

$$\int_{S^2 \times S^1} \frac{a da}{4\pi^2} \longmapsto \int_{S^2 \times S^1} \frac{a da}{4\pi^2} + \frac{1}{\pi} \underbrace{\int_{S^1} \epsilon^{(1)}}_{\in 2\pi\mathbb{Z}} \underbrace{\int_{S^2} \frac{da}{2\pi}}_{= n}$$

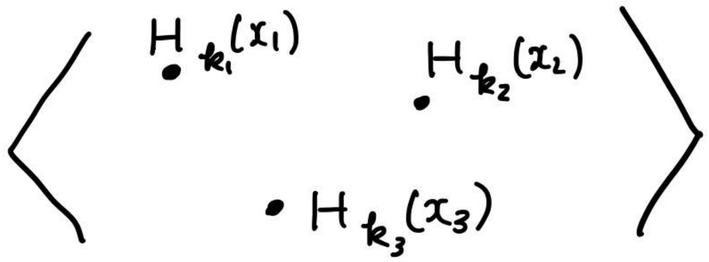
$$= \int_{S^2 \times S^1} \frac{a da}{4\pi^2} + 2n\mathbb{Z}.$$

$\Longrightarrow \int_{S^2 \times S^1} \frac{a da}{4\pi^2} = k$ is well-defined only in $\mathbb{Z}_{2n}$, i.e. $k \sim k + 2n$.

[cf. Pontrjagin '41]

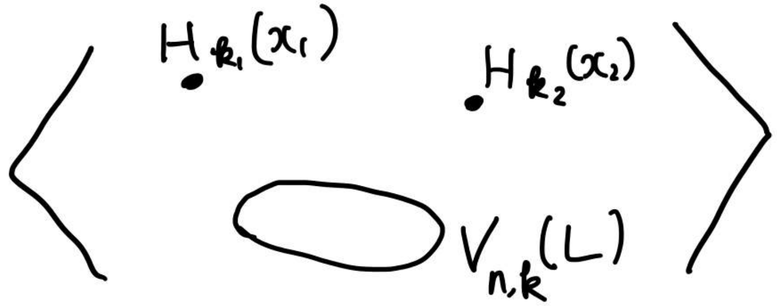
Is the Hopfion symmetry $U(1)$ or $\mathbb{Z}_2$?

Let's evaluate correlation functions in a compact spacetime.



$$\neq 0 \Rightarrow k_1 + k_2 + k_3 + \cdots = 0.$$

$U(1)$ conservation law



$$\neq 0 \Rightarrow k_1 + k_2 + \cdots + k = 0 \mod 2n.$$

$\mathbb{Z}_{2n}$ conservation law.

- Without $V(L)$, the Hopfion number conserves as if there is a $U(1)$ symmetry.
- With $V(L)$, the conservation law reduces to that of $\mathbb{Z}_2$.

Which is the symmetry group? Or, is it something else?

Generalized Symmetry in QFTs

Generalized Symmetry = Topological Operators

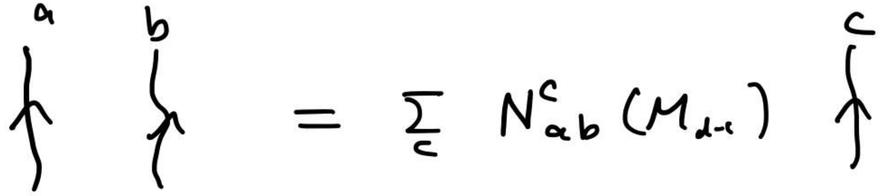
For continuous symmetry,

$Q(M_{d-1}) = \int_{M_{d-1}} j$ is invariant under any (topological) continuous deformation of M_{d-1}.

In conventional symmetry, those topological operators obey group structures.

However, this turns out to be too restrictive to explore QFTs.

Non-invertible symmetry (or Categorical symmetry)



$$= \sum_c N_{ab}^c(M_{d-1})$$

Fusion rule of symmetry defects can be quite general.

(2d CFTs : Verlinde '88, Bhadwaj, Tachikawa '17, Thorngren, Wang '19, ...

Higher dims (Since '21) : Nguyen, YT, Ünsal ; Heidenreich, McNamara, Reece, Rudelius, Valenzuela; Koide, Nagoya, Yamaguchi ; Choi, Cordova, Hsin, Lam, Shao ; Kaidi, Ohmori, Zhong ; ...)

Topological operators and TQFTs

One of useful methods to find "unconventional" sym. :(cf. Choi, Lam, Shao '22; Cordova, Ohmori '22)

1. Prepare a TQFT
2. Put it on a submanifold with a topological coupling to dynamical fields.

(As every ingredient is topological, this operator is manifestly topological. We should check if it acts nontrivially to local operators.)

In our case,

1. We prepare the level-N $U(1)$ CS theory $\int \mathcal{D}b\, e^{i\frac{N}{4\pi}\int b\,db}$

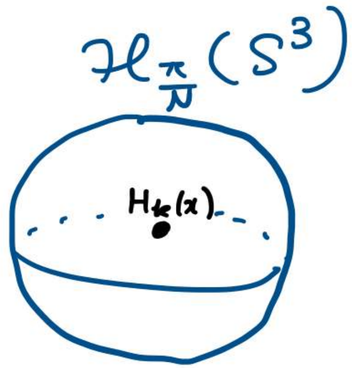
2. The Hopfion symmetry operator is then defined as

$$\mathcal{H}_{\frac{\pi}{N}}(M_3, da) = \int \mathcal{D}b \exp\left(i\frac{N}{4\pi}\int_{M_3} b\wedge db + i\frac{1}{2\pi}\int_{M_3} b\wedge da \right).$$

Let's check how this operator acts on $H_k(x)$ and $V_{n,k}(L)$.

Hopfion op. vortex op.

Action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$ on $H_k(x)$



To evaluate the action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$, we can set $M_3 = S^3$ that surrounds x.

$$\mathcal{H}_{\frac{\pi}{N}}(S^3) = \int \mathcal{D}b \; e^{i\frac{N}{4\pi}\int b\,db + i\frac{1}{2\pi}\int b\,da}$$

$$= \int \mathcal{D}b \; e^{i\frac{N}{4\pi}\int_{S^3}(b+\frac{a}{N})\,d(b+\frac{a}{N})} \cdot e^{-i\frac{\pi}{N}\int_{S^3}\frac{a\,da}{4\pi^2}}$$

On S^3, $U(1)$ bundle is trivial.

$$\propto e^{-i\frac{\pi}{N}\int_{S^3}\frac{a\,da}{4\pi^2}}$$

This shows that

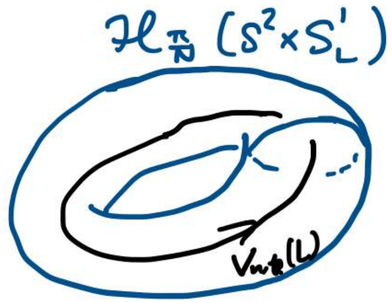
$$\langle \mathcal{H}_{\frac{\pi}{N}}(S^3)\; H_k(x) \cdots \rangle \propto e^{-i2\frac{\pi}{N}k} \langle H_k(x) \cdots \rangle .$$

and $\mathcal{H}_{\frac{\pi}{N}}(S^3)$ detects the Hopfion charge of $H_k(x)$.

Since N can be arbitrary, $\{\mathcal{H}_{\frac{\pi}{N}}(S^3)\}_{N\geq 1}$ determines k as an integer.

$\Rightarrow$ Recovery of $U(1)$-like selection rule

Action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$ on $V_{n,k}(L)$



Next, we set $M_3 = S^2 \times S^1_L$ to evaluate its action on $V_{n,k}(L)$.

$$\mathcal{H}_{\frac{\pi}{N}}(S^2\times S^1) = \int \mathcal{D}b\, e^{i\frac{N}{4\pi}\int b\,db + i\frac{1}{2\pi}\int b\,da}$$

$\int_{S^1} b$ $\quad \int_{S^2}\frac{db}{2\pi}$ $\quad \int_{S^2}\frac{da}{2\pi}$

$$= \sum_m (\cdots) \int_0^{2\pi} d\beta\, e^{i\underset{\sim}{\beta}(N\underline{\underline{m}} + \underline{\underline{n}})}$$

U(1) bundle over $S^2\times S^1$ can be nontrivial.

$= 0$ unless $Nm+n=0$.

$$= \begin{cases} e^{-i\frac{\pi}{n}k} & \text{for } N=n. \\ 0 & \text{if } N \text{ is not a divisor of } n. \end{cases}$$

When the symmetry looks to be reduced to $\mathbb{Z}_{2n}$ by the presence of vertex operators, $\mathcal{H}_{\frac{\pi}{N}}(S^2\times S^1)$ acts nontrivially only if it fits the periodicity of $\mathbb{Z}_{2n}$.

Moreover, $\mathcal{H}_{\frac{\pi}{n}}(S^2\times S^1)$ captures the Hopfion charge of $V_{n,k}(L)$ in mod $2n$.

For 4d $\mathbb{C}P^1$ σ-model,

the Hopfion symmetry associated with $\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ is neither $U(1)$ nor $\mathbb{Z}_2$.

The correct symmetry generator is given by

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \; e^{i\frac{N}{4\pi}\int_{M_3} b\wedge db + i\frac{1}{2\pi}\int_{M_3} b\wedge da},$$

and the fusion rule is controlled by those of 3d TQFTs.

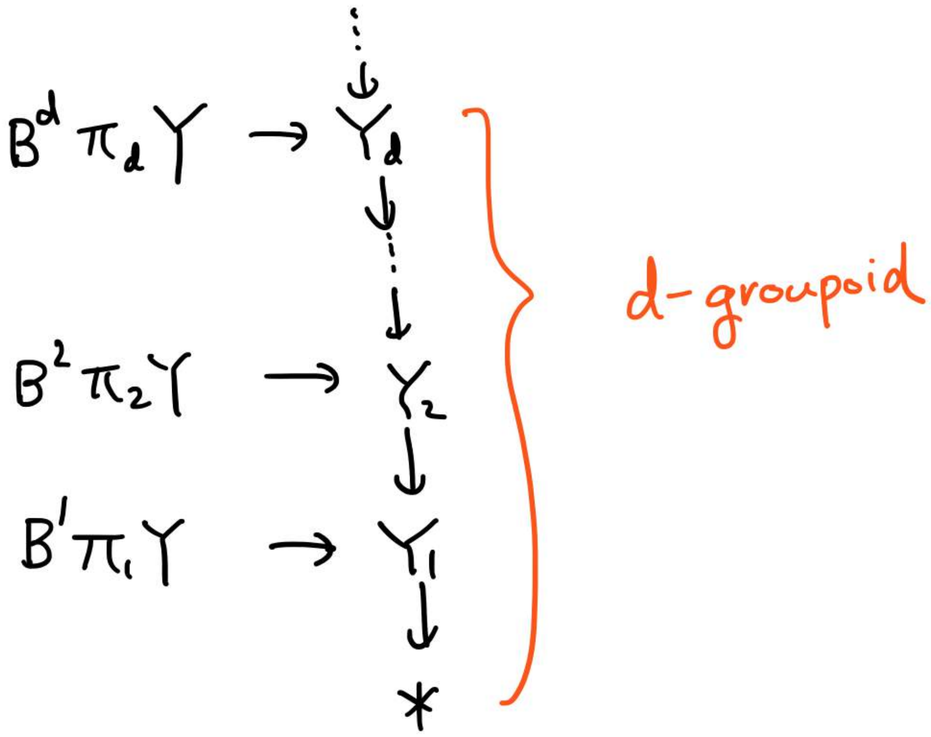
There is an invertible $\mathbb{Z}_2$ subgroup, generated by

$$\mathcal{H}_{\pi}(M_3) = e^{i\pi\int_{M_3}\frac{a\,da}{4\pi^2}} \in \mathbb{Z}_2 \left(\simeq \widetilde{\Omega}^{Spin}_3(\mathbb{C}P^1)\right).$$

Mathematical structure of solitonic symmetry

Let's formally consider a nonlinear σ-model w/ a target space Y.

It's convenient to use the Postnikov system $Y \to \cdots \to Y_d \to Y_{d-1} \to \cdots \to Y_0 \simeq *$:



Solitonic symmetry "=" magnetic symmetry of "d-groupoid gauge theory".

Two origins of non-invertibility

If $\pi_{1,\cdots,k-1} Y \simeq 0$ and $\pi_k Y$ is nontrivial, Hurwicz theorem shows

$$\pi_k Y \simeq H_k(Y) \Rightarrow \text{invertible } \mathrm{Hom}(\pi_k Y, U(1)) \text{ symmetry.}$$

For $\pi_{\ell>k} Y$, the symmetry can be non-invertible.

- Non-invertibility due to $\pi_1 Y$-action.

$\pi_1 Y \curvearrowright \pi_{k>1} Y \Rightarrow$ Outer automorphism is gauged

$\Rightarrow$ "cosine symmetry" [Nguyen, YT, Ünsal '21, Heidenreich, McNamara, et al. '21; Arias-Tamargo, Rodriguez-Gomez '22, ...]

- Non-invertibility due to non-trivial extension in the Postnikov tower:

$$B^n \pi_n Y \longrightarrow Y_n \longrightarrow Y_{n-1} \simeq \mathrm{Rep}^{n-1}(Y)$$

$B^n \pi_n Y$: nD top. func. non trivial on S^n: conventional treatment

$Y_{n-1} \simeq \mathrm{Rep}^{n-1}(Y)$: nD top. func. trivial on S^n: condensation / higher gauging.

Summary

- Topological conservation law for solitons is not fully characterized by homotopy groups.
- 4d $\mathbb{C}P^1$ σ-model is carefully examined.

$$\begin{cases} \pi_2(\mathbb{C}P^1) \simeq \mathbb{Z} & \Rightarrow \ U(1) \text{ 1-form symmetry for vortex.} \\ \pi_3(\mathbb{C}P^1) \simeq \mathbb{Z} & \not\Rightarrow \ U(1) \text{ symmetry for Hopfion.} \end{cases}$$

$$\left\langle \begin{array}{c} H_{k_1}(x_1) \quad H_{k_2}(x_2) \\ \bullet \qquad\quad \bullet \\ \\ \bigcirc \\ V_{n,k}(L) \end{array} \right\rangle \neq 0 \Rightarrow k_1 + k_2 + \cdots + k = 0 \mod 2n.$$

- The symmetry generators are given by 3d TQFT partition functions

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \ \exp\left(i \int_{M_3} \frac{N}{4\pi} b \, db + i \int_{M_3} \frac{1}{2\pi} b \, da \right).$$

$\Rightarrow$ Noninvertible Solitonic Symmetry

- Non-invertible solitonic symmetry $\approx$ Magnetic symmetry of higher-group gauge theory